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A TEST FOR LEGITIMATE DECKS

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Bondy and Hemminger (Graph reconstruction—a survey, J. Graph Theory 1 (1977) 227–268) give three necessary conditions namely subgraph condition, degree sequence condition and symmetric array condition (of Randic) for Legitimate decks. We have strengthened the degree sequence condition (SDSC) and extended the symmetric array condition (ESAC). We prove that both these are necessary conditions for legitimate decks and ESAC implies SDSC and symmetric array condition. ESAC gives a graph G^* which has the same degree sequence as the prospective graph determined by the deck and the point deletions of G^* have the same degree sequence as the given cards. Examples of illegitimate decks satisfying ESAC, legitimate decks giving G^* with point deletions different from the given deck, deck giving more than one (non isomorphic) G^* etc. are given. We develop this ESAC to a test to determine whether a deck is legitimate or not. Also we point out how the failure of reconstruction conjecture (if it is so) for $(p-1)$-point graphs can hamper the characterization of legitimate decks of p-point graphs. We extend these results to colored graphs and digraphs also.

1. Introduction

In the first four sections, we consider simple undirected graphs without loops and multiple edges. Bondy and Hemminger [1] give the following necessary conditions for a deck $(G_i \mid 1 \leq i \leq n)$ to be legitimate.

(1) For every graph F with $v(F) < n$,

$$n - v(F) \,\Big|\, \sum_{i=1}^{n} s(F, G_i)$$

and

$$s(F, G_j) \leq \sum_{i=1}^{n} s(F, G_i)/(n - v(F)), \quad 1 \leq j \leq n$$

where $v(F)$ is the number of points in F and $s(F, G_i)$ is the number of times F occurs in G_i as an induced subgraph.

(2) The "symmetric array" condition [4, 5]:

The vertex deleted subgraphs of the cards G_i can be arranged in a symmetric $n \times n$ array so that for $1 \leq i \leq n$, the vertex deleted subgraphs of G_i appear as the nondiagonal entries of row i.

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(3) The degree sequence of the cards are compatible (in an obvious way).

The collection of vertex deleted subgraphs of a graph is called "the deck of the graph". The vertex deleted subgraphs are sometimes called "cards". We use the terminology in Harary [2].

2. Strengthened degree sequence condition (SDSC)

Let the graph G have degree sequence $d_1, d_2, \ldots, d_n$. If a point of degree d_i is removed, the degree sequence of the resulting graph can be obtained from this degree sequence by dropping the entry d_i and reducing d_i of the remaining entries each by one. So the point deletions of G can be arranged as G_i, $i=1$ to n such that the degree sequence of G_i is obtainable from that of G by deleting d_i and reducing d_i of the remaining entries each by one. In fact we can say more about the degree sequences of the point deletions.

Result 1. *Let G be a graph. The total number of points of degree r in all its point deletions is $(r+1)K_{r+1}+(n-r-1)K_r$ where K_r is the number of points of degree r in G.*

Proof. Each point of degree $r+1$ becomes a point of degree r in exactly $r+1$ cards. So the contribution of points of degree $r+1$ in G to the sum considered is $(r+1)\,K_{r+1}$.

Each point of degree r will remain as a point of degree r in n-$(r+1)$ cards. Thus the contribution of points of G of degree r to the sum considered is $(n-r-1)\,K_r$.

Since these are the only two types of points which give points of degree r in the cards, the result follows.

This condition, together with the degree sequence condition mentioned above are necessary for legitimate decks and we call them together as *strengthened degree sequence condition* (*SDSC*). Example 1 given in next section satisfies the degree sequence condition but not SDSC.

3. Extended symmetric array condition (ESAC)

(a) Replace each entry of the $n\times n$ symmetric array in (2) by the number of lines in it. Let this array be denoted by A and let a_{ij} denote its entry in the ith row jth column.

(b) Form a new array B with entries b_{ij} where

$$b_{ij} = (\text{number of lines in } G_i) - a_{ij}.$$

(c) Form a new array C with entries c_{ij} where

$$c_{ij} = p_j - b_{ij}$$

where p_j is the degree of the point j in G calculated from the deck.

If there is a symmetric array mentioned in (2) above such that C is symmetric with entries 0 or 1, the deck G_i, $i=1$ to n is said to satisfy the *extended symmetric array condition* (*ESAC*).

Make C into a symmetric binary matrix by filling the diagonal entries with zeros. Let G^* denote the graph having this as the adjacency matrix.

Result 2. *ESAC is a necessary condition for legitimate decks. Also each graph having the given legitimate deck can be realised as G^* for a suitable symmetric array.*

Proof. Let G be a graph having the given deck G_i, $i=1$ to n as point deletions. Give the labels 1 to n to the points of G so that $G_i = G - i$. Give the labels inherited from G to the points of each G_i. In the ijth place of the array put the graph $G_i - j$, thus getting a symmetric array. When we construct G^*, i adj j in $G \langle = \rangle\ b_{ij} = \deg_G j - 1$ and $b_{ji} = \deg_G i - 1$

$$\langle = \rangle c_{ij} = 1 \quad \text{and} \quad c_{ji} = 1.$$

Similarly i not adjacent to j in $G \langle = \rangle\ c_{ij} = 0$ and $c_{ji} = 0$.

Therefore C is a symmetric binary array and v_i adj v_j in G^* iff i adj j in G. Thus $G \cong G^*$ under $i \to v_i$.

Note. Here $G^* - v_i \cong G_i$, $1 \leq i \leq n$, and in this isomorphism v_j of $G^* - v_i$ is mapped to a point of G_i whose removal is isomorphic to the entry in the ijth place of the symmetric array. Also Result 2 implies that graphs for which all possible symmetric arrays give the same array C in (c) are reconstructible.

Result 3. *The graph G^* has the same degree sequence as the one determined by the deck.*

Proof. Let q_i be the number of lines in G_i. Let q be the total number of lines in G and p_i be the degree of the point i in G. q and p_i are determined by the deck.

In B, b_{ij} is the degree of the point chosen as the jth point in G_i. Therefore in B, sum of the ith row entries is $2q_i = 2(q - p_i)$. Now sum of the ith row entries of C is

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq 1}}^{n} (p_j - b_{ij}) &= \sum_{\substack{j=1\\ j\neq 1}}^{n} p_j - \sum_{\substack{j=1\\ j\neq 1}}^{n} b_{ij} \\ &= (2q - p_i) - 2(q - p_i) \\ &= p_i\end{aligned}$$

Since C is (modified to) the adjacency matrix of G^*, we get that the ith point of G^* has degree p_i.

Result 4. *The degree sequence of the* card G_i, $i=1$ *to* n *is same as the degree sequence of* $G^* - v_i$ *where* v_i *is the* ith *point of* G^*.

Proof. $c_{ij} = p_j - b_{ij}$. Therefore $b_{ij} = p_j - c_{ij}$. If v_i adj v_j in G^*, $c_{ij} = 1$ and hence $b_{ij} = p_j - 1$. If v_i not adj v_j in G^*, $c_{ij} = 0$ and hence $b_{ij} = p_j$.

Thus b_{ij}, $j = 1$ to n, $j \neq i$ gives the degree sequence of $G^* - v_i$. But from (b), b_{ij}, $j = 1$ to n, $j \neq i$ denotes the degree sequence of G_i. Hence the result.

Because of Result 4, we see that the ESAC implies the SDSC (in fact any condition on degrees arising from legitimacy). It is easy to prove that whenever symmetric array condition and degree sequence condition are true, the array C obtained is symmetric and the sum of the rth row entries of C is p_r but the entries need not be 0 and 1 (Example 1). Also we can give examples to establish the following.

(1) and the symmetric array condition together does not imply ESAC (Example 2).

ESAC is not sufficient for legitimacy (Example 3).

G^* obtained from a legitimate deck need not have point deletions isomorphic to the cards of the deck (Example 4).

A deck (legitimate/illegitimate) can give more than one G^* (Examples 4 and 3).

Legitimate and illegitimate decks giving the same G^* with point deletions different from the given deck (Examples 4 and 3).

The deck of a connected graph can give a disconnected graph as G^* and vice versa (Examples 4 and 5).

G^* obtained from the deck of G can have different chromatic number and edge chromatic number from that of G (Example 6).

Example 1. The deck consisting of one copy of $2K_1 \cup K_2$ and four copies of $K_1 \cup K_{1,2}$ satisfies degree sequence condition and give 2, 1, 1, 1, 1 as the degree sequence of its parent graph.

$2K_1 \cup K_2$		A	A	B	B
$K_1 \cup K_{1,2}$	A		B	B	C
$K_1 \cup K_{1,2}$	A	B		C	B
$K_1 \cup K_{1,2}$	B	B	C		A
$K_1 \cup K_{1,2}$	B	C	B	A	

$A = \bar{K}_3 : B = K_2 \cup K_1 : C = K_{1,2}$

is a symmetric array. But this deck does not satisfy ESAC (for any symmetric array).

Example 2. The deck consisting of 8 copies of $K_{1,6}$. This satisfies (1) (proved in Jackson [3]) and symmetric array condition but not ESAC.

Example 3. The deck consisting of 8 copies of $C_5 \cup K_2$. The parent graph (if any) must be regular of degree 2. The vertex deleted subgraphs of $C_5 \cup K_2$ are A, A, A, A, A, B, B where A stands for $P_4 \cup K_2$ and B stands for $C_5 \cup K_1$ (P_n is the path with n points). The symmetric array

$$\begin{array}{cccccccc}
 & & & & & & B & B \\
 & & & & & & B & B \\
 & & & & B & B & & \\
 & & & & B & B & & \\
 & & B & B & & & & \\
 & & B & B & & & & \\
B & B & & & & & & \\
B & B & & & & & &
\end{array}$$

with vacant non diagonal places filled with A gives $2C_4$ as G^* whereas the symmetric array

$$\begin{array}{cccccccc}
 & & & & & B & & B \\
 & & & & B & & B & \\
 & & & B & & B & & \\
 & & B & & B & & & \\
 & B & & B & & & & \\
B & & B & & & & & \\
 & B & & & & & & B \\
B & & & & & & B &
\end{array}$$

with vacant non diagonal places filled with A gives C_8 as G^*. However, the deck is illegitimate as it does not satisfy (1).

Example 4. The vertex deleted subgraphs of $2C_4$ are $C_4 \cup P_3$ (8 copies). The point deletions of each of these are A, A, A, A, B, B, C, where $A = 2P_3$, $B = C_4 \cup P_2$ and $C = C_4 \cup 2K_1$. The symmetric array

$$\begin{array}{cccccccc}
 & & & C & & B & & B \\
 & & C & & B & & B & \\
 & C & & B & & B & & \\
C & & B & & B & & & \\
 & B & & B & & & & C \\
B & & B & & & & C & \\
 & B & & & & C & & B \\
B & & & & C & & B &
\end{array}$$

with vacant non diagonal places filled with A gives C_8 as G^*.

Example 5. The vertex deletions of C_8 give $2C_4$ as G^* under suitable symmetric array.

Example 6. The deck of $2C_3$ give C_6 as G^* under suitable symmetric array.

We do not know examples to show the following:

$(1)+$ESAC does not imply legitimacy.

Symmetric array$+$SDSC does not imply ESAC.

$(1)+$symmetric array$+$SDSC does not imply legitimacy.

However, condition (1) above and ESAC together are unlikely to be proved to be sufficient conditions for the legitimacy of the deck, without assuming the truth of the Reconstruction Conjecture for $(n-1)$-point graphs. Suppose E and F are two non-isomorphic graphs with $n-1$ points, both having the same deck. Let G be a connected supergraph of E having n points so that E is an induced subgraph of G. Now the deck of G has E as a card. For this deck, condition (1) and ESAC are true since it is legitimate. So, for a particular symmetric array, $G^* \cong G$ under the isomorphism $v_i \to i$. Now in this deck, replace E by F. For the new deck also, the same symmetric array satisfies the ESAC.

We have $s(H, E) = s(H, F)$ whenever $v(H) < n-1$, since both E and F have the same deck and the deck uniquely determines these numbers. When $v(H) = n-1$, condition (1) is trivially true. Thus the new deck satisfies condition (1) also. It is highly unlikely that the new deck is legitimate as it differs from the deck of a connected graph in just one card. In the digraph case discussed in Section 5, such decks can actually be constructed even though such a deck does not automatically satisfy ESAC.

4. Test for legitimacy

The ESAC can be used to test whether the given deck is legitimate or not. If the given deck satisfies no symmetric arrays, it is illegitimate. Otherwise, find those symmetric arrays that satisfy ESAC. If none of them satisfy ESAC, the deck is illegitimate. Among the arrays that satisfy ESAC, if there is one whose G^* is such that $G^* - v_i \cong G_i$, $i = 1$ to n the deck is legitimate, otherwise illegitimate. (Once we denote the point deletions of G_i with alphabets as in the examples above, the search for symmetric arrays becomes easier.)

A symmetric array
SS

- ESAC is satisfied
  - $G^* - v_i \cong G_i$ for $i = 1$ to n. Deck is legitimate. STOP.
  - $G^* - v_i \not\cong G_i$ for at least one i. Repeat the test starting from MM.
- ESAC is not satisfied.
  MM
  - There exists a symmetric array not tested before. Do the test with this array starting from SS.
  - Every symmetric array is already tested. The deck is illegitimate. STOP.

5. Digraphs and colored graphs

We can extend the necessary conditions and test to digraphs also. With each point v of a digraph D we can associate a triple (r, s, k) called degree triple where v is incident with r unpaired outarcs, s unpaired in-arcs and k symmetric pair of arcs. Here we assume that D_i, $i=1$ to n are the given digraphs, each having $n-1$ points and $n>4$.

Condition (1) given in Section 1 is a necessary condition for legitimate decks of digraphs also.

(DTSC): A deck is said to satisfy the *degree triple sequence condition* (DTSC) if for each i the degree triple sequence of the card D_i is obtainable from that of the prospective digraph by deleting the triple (r_i, s_i, k_i) and subtracting $(0,1,0)$ from exactly r_i of the remaining $n-1$ triples, subtracting $(1,0,0)$ for exactly s_i of the remaining $(n-1-r_i)$ triples and subtracting $(0,0,1)$ from exactly k_i of the remaining $(n-1-r_i-s_i)$ triples. (Subtraction is co-ordinate wise.)

(SDTSC): If a deck satisfies DTSC and the total number of points having degree triple (i, j, k) in all the cards is

$$(i+1)(i+1, j, k)^* + (j+1)(i, j+1, k)^* + (k+1)(i, j, k+1)^* + (n-i-j-k-1)(i, j, k)^*$$

where $(i, j, k)^*$ is the number of points of degree triple (i, j, k) in D, we say that the deck satisfies *strengthened degree triple sequence condition* (SDTSC). It is obvious that SDTSC is a necessary condition.

When a point is removed from a digraph, we get a subdigraph (card). The degree triple of this point is called the *natural degree triple* of this card and this way of associating degree triples with cards of a given digraph is called the *natural association of degree triples*. Even though the subdigraphs $D_i = D - v_i$ determine the degree triple sequence of D uniquely, these degree triples can be associated with the cards in more than one way satisfying SDTSC. In the digraph D given in Fig. 1 let $D_1 = D - v_1$ and $D_2 = D - v_2$. v_1 and v_2 have degree triples $(1,2,1)$ and $(2,1,1)$

D :

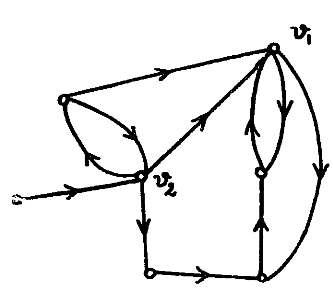

Fig. 1.

respectively and $D_1 \not\cong D_2$. The association of $(1,2,1)$ with D_2, $(2,1,1)$ with D_1 and the natural degree triples with all other cards give an association of degree triples different from the natural association, but satisfying SDTSC.

(ESAC): Take a symmetric array mentioned in (2) of Section 1. Take a particular association of degree triples with the cards D_i satisfying DTSC and proceed as follows:

Replace the digraph in the ijth place of the array with a degree triple b_{ij} so that b_{ij} for each i give the natural association for the subdigraphs of D_i getting array B whose non diagonal places are filled with triples.

Form a new array C with entries e_{ij} where $e_{ij} = p_j - b_{ij}$ where p_j is the triple associated with card D_j in the chosen association of degree triples.

If there is a symmetric array and associations of degree triples such that the new array C obtained satisfies (i) each entry of C is either $(0,0,0)$ or $(1,0,0)$ or $(0,1,0)$ or $(0,0,1)$ and (ii) $e_{ij} + e_{ji} = (0,0,0)$ or $(1,1,0)$ or $(0,0,2)$, we say that the deck satisfies *extended symmetric array condition* (ESAC). In this case we can construct a digraph D^* with vertices $v_1, v_2, \ldots, v_n$ by drawing an arc from v_i to v_j whenever e_{ij} is $(0,1,0)$ or $(0,0,1)$.

The following results similar to Results 2, 3, and 4 can be proved in a similar manner.

Result. *ESAC is a necessary condition for legitimate decks of digraphs. Also each digraph having the given legitimate deck can be realised as D^* for a suitable symmetric array and a suitable collection of association of degree triples.*

Result. *The digraph D^* has the same degree triple sequence as the one determined by the given deck. The degree triple sequence of the* card D_i *is same as the degree triple sequence of $D^* - v_i$.*

Here again we see that SDTSC is implied by ESAC. Since graphs are special type of digraphs, the examples in Section 3 are applicable here also. In the digraph case, SDTSC and symmetric array condition together does not imply legitimacy. The deck of the digraph D in Fig. 2 with the card $D - v$ replaced by T is an example for this.

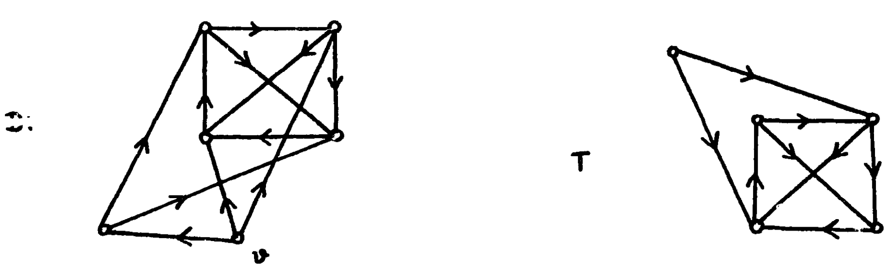

Fig. 2.

Test for legitimacy. The test for legitimacy given for graphs applies for digraphs also. But here, for each symmetric array we have to perform the test for each association of degree triples to D_i, $i = 1$ to n satisfying DTSC and for each natural association of degree triples to the point deletions of each D_i. Note that once the point deletions of D_i are fixed in places in the symmetric array, it may be possible to give more than one natural association of degree triples which have different effects on C. As an example, if D_i is P_r (directed path with r points), then P_{r-1} will occur in two places in the ith row of the symmetric array, say ij and ik. Taking $b_{ij}=(1,0,0)$ and $b_{ik}=(0,1,0)$ will not have the same effect on C as taking $b_{ij}=(0,1,0)$ and $b_{ik}=(1,0,0)$ even though both can be extended to the natural association. The deck of the digraph D in Fig. 2 with the card $D-v$ replaced by T can be proved to be illegitimate and the process is a good illustration of the test as it involves nontrivial possibilities in almost all stages.

There are non reconstructible tournaments on 2^n+2^m vertices for all m and n, not both zero (due to Stockmeyer). So constructions of decks as mentioned at the end of Section 3 aimed at showing "(1)+ESAC does not imply legitimacy" can be done. To decide the legitimacy of such decks constructed we have to use the test given above.

In the case of colored graphs also (a graph together with a partition of the vertex set into color classes: two colored graphs are isomorphic if there is an isomorphism between them as graphs such that corresponding vertices have the same color) a similar test for legitimacy of deck can be given. In this case G^* has the same degree array as the colored graph determined by the deck and the point deletions of G^* have the same degree array as the given cards. For edge deleted cards, legitimacy via their line graphs can be tested.

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